

Semiclassical Theory for Universality in Quantum Chaos with Symmetry Crossover

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Abstract. We address the quantum-classical correspondence for chaotic systems with a crossover between symmetry classes. We consider the energy level statistics of a classically chaotic system in a weak magnetic field. The generating function of spectral correlations is calculated by using the semiclassical periodic-orbit theory. An explicit calculation up to the second order, including the non-oscillatory and oscillatory terms, agrees with the prediction of random matrix theory. Formal expressions of the higher order terms are also presented. The nonlinear sigma (NLS) model of random matrix theory, in the variant of the Bosonic replica trick, is also analyzed for the crossover between the Gaussian orthogonal ensemble and Gaussian unitary ensemble. The diagrammatic expansion of the NLS model is interpreted in terms of the periodic orbit theory.

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1. Introduction

The quantum-classical correspondence in a chaotic system is a longstanding problem. In a quantum system, chaos does not exist in the sense of the "sensitivity to the initial conditions", while in a classical system, it exists and a characterization is given by a positive Lyapunov exponent. A signature of chaos in a quantum system can be seen in the energy level statistics, which shares the same behavior with the eigenvalue statistics of random matrices [1]. According to a well-known conjecture by Bohigas, Giannoni, and Schmit [2], the energy level statistics in classically chaotic systems with and without time-reversal symmetry are universally described by the Gaussian orthogonal ensemble (GOE) and Gaussian unitary ensemble (GUE), respectively.

Recently a justification of this conjecture has been given for the spectral form factor, in terms of the semiclassical periodic-orbit theory[3]. It is based on Berry's diagonal approximation [4] for the first order term and the periodic-orbit pairs with an encounter [5] for the second order term. Higher order terms are also calculated by using the periodic pairs with many encounters. If a very weak magnetic field is applied to a classically chaotic system, a crossover between the GOE and GUE universality classes is realized. The level statistics in this domain is described by Pandey and Mehta's two matrix model [6]. The periodic-orbit theory is also applicable to explain the behavior in this GOE-GUE crossover domain [7, 8, 9, 10, 11]. In this approach, a stochastic behavior of the periodic orbits plays a crucial role. Similar ideas also give evaluations of GUE-GUE[11, 12], GOE-GOE[12], GSE-GSE[13] and GOE-GSE[13] form factors.

Moreover a recent progress on this topic has made it possible to reproduce the oscillatory terms in the expansion of the GOE and GUE spectral correlations [14]. In this new scheme, a periodic orbit theory is constructed for the generating function which yields the spectral correlation. The generating function is defined as

$$Z(\epsilon_A, \epsilon_B, \epsilon_C, \epsilon_D) = \left\langle \frac{\det(E_C^+ - H) \det(E_D^- - H)}{\det(E_A^+ - H) \det(E_B^- - H)} \right\rangle. \quad (1)$$

where H is the Hamiltonian. Here $E_{A,C}^+$ and $E_{B,D}^-$ are given by $E_{A,C}^+ = E + \frac{\epsilon_{A,C}^+}{2\pi\bar{\rho}}$ and $E_{B,D}^- = E + \frac{\epsilon_{B,D}^-}{2\pi\bar{\rho}}$ with the local average of the level density $\bar{\rho}$, where the offsets from the center energy E are taken with an imaginary part as $\epsilon^\pm = \epsilon \pm i\gamma$ ($\gamma > 0$) to ensure the convergence. The generating function (1) yields the spectral correlation function $C(\epsilon)$ as

$$\begin{aligned} \left. \frac{\partial^2 Z}{\partial \epsilon_A \partial \epsilon_B} \right|_{||} &= \frac{1}{(2\pi\bar{\rho})^2} \text{Tr} \left(\frac{1}{E + \frac{\epsilon}{2\pi\bar{\rho}} + i\gamma - H} \right) \text{Tr} \left(\frac{1}{E - \frac{\epsilon}{2\pi\bar{\rho}} - i\gamma - H} \right) \\ &= \frac{1}{2} C(\epsilon) + \frac{1}{4}, \end{aligned} \quad (2)$$

where the symbol $||$ denotes the identification $\epsilon_A^+ = \epsilon_C^+ = \epsilon^+$, $\epsilon_B^- = \epsilon_D^- = -\epsilon^+$. A crucial idea of the new scheme is counting all the diagrams in terms of the pseudo orbits, which correspond to the terms appearing in a field theoretical treatment of random matrices.

In this paper, we extend this new scheme to the case with a weak magnetic field, which describes the GOE-GUE crossover. We combine the ideas of Refs.[14] and [10], and compare the results of periodic-orbit and random matrix theories. The random matrix theory prediction for the generating function is known to be [15],

$$Z(\epsilon_A, \epsilon_B, \epsilon_C, \epsilon_D) = Z^{(1)} + Z^{(2)}, \quad (3)$$

where

$$\begin{aligned} Z^{(1)} &= e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \frac{(\epsilon_A - \epsilon_D)(\epsilon_C - \epsilon_B)}{(\epsilon_A - \epsilon_B)(\epsilon_C - \epsilon_D)} [1 + Z_{\text{off}}^{(1)}], \\ Z_{\text{off}}^{(1)} &= \frac{\epsilon_{AC}\epsilon_{BD}}{4} \int_1^\infty dp p^{-1} e^{i(p-1)\epsilon_{AB}/2 - 2sp^2} \int_0^1 dq q e^{-i(q-1)\epsilon_{CD}/2 + 2sq^2} \\ &= \frac{2}{i(\epsilon_{AB} + i8s)} - \frac{2}{i(\epsilon_{AD} + i8s)} - \frac{2}{i(\epsilon_{CB} + i8s)} + \frac{2}{i(\epsilon_{CD} + i8s)} \\ &\quad + 16s\epsilon_{AC}\epsilon_{BD} \left[\frac{1}{(\epsilon_{CD} + i8s)^2(\epsilon_{AD} + i8s)(\epsilon_{CB} + i8s)} \right. \\ &\quad \left. - \frac{1}{(\epsilon_{AB} + i8s)^2(\epsilon_{AD} + i8s)(\epsilon_{CB} + i8s)} \right] + \dots \end{aligned} \quad (5)$$

with $\epsilon_{\alpha\beta} = \epsilon_\alpha - \epsilon_\beta$. The function $Z^{(2)}$ is obtained from the function $Z^{(1)}$ by exchanging ϵ_C and ϵ_D . The parameter s controls the symmetry: the GOE (GUE) limit corresponds to $s \rightarrow 0$ ($s \rightarrow \infty$). In this paper, we explicitly calculate the generating function up to the second order using the periodic-orbit theory, and confirm the agreement with the above prediction. Moreover, in order to see the correspondence of the higher order terms, we examine the nonlinear sigma (NLS) model in the crossover domain, and interpret the diagrammatic expansion of the NLS model in terms of periodic pairs.

2. Semiclassical Expression of the Generating Function

2.1. Magnetic Action

We consider a bounded quantum system with f degrees of freedom whose corresponding classical dynamics is chaotic. We assume that the system has a gauge potential due to a magnetic field B . Let us denote the energy by E and each phase space point by a $2f$ dimensional vector $\mathbf{x} = (\mathbf{q}, \mathbf{p})$, where f dimensional vectors \mathbf{q} and \mathbf{p} specify the position and momentum, respectively. In the semiclassical limit $\hbar \rightarrow 0$, the trace formula for the energy-level density implies

$$\text{Tr} (E^+ - H)^{-1} \sim -i\pi\bar{\rho} - \frac{i}{\hbar} \sum_a T_a F_a e^{\frac{i}{\hbar}[S_a(E^+) + \Theta_a(B)]}, \quad (6)$$

where H is the Hamiltonian. The first term in the right hand side gives the local average of the level density $\bar{\rho}$ and the second term describes the fluctuation around the average. The local average of the level density is equal to the number of Planck cells inside the energy shell

$$\bar{\rho} = \frac{\Omega(E)}{(2\pi\hbar)^f}, \quad (7)$$

where $\Omega(E)$ is the volume of the energy shell. The function S_a is the classical action (including the Maslov phase) of the periodic orbit a with a period $T_a = dS_a/dE$, F_a is the stability amplitude and Θ_a is the magnetic action.

We consider the following two possible situations. One is the case that the system has an Aharonov-Bohm type gauge potential, where the gauge potential exists although the magnetic field is not directly applied to the system. Then the classical dynamics is time-reversal invariant, while the quantum counterpart is not. Another situation is the case that the magnetic field is directly applied to the particle, but it is sufficiently weak such that the cyclotron radius is much larger than the system size and thus the presence of the magnetic field does not significantly change $\Omega(E)$. In this case the impact of the magnetic field on the orbits can be neglected. In both cases the magnetic field still influences the behavior of the system through its contribution to the phase in (6). Time-reversed classical dynamics must be taken into account to derive the form factor in the crossover domain; it even becomes nontrivial to derive the form factor in the GUE limit.

The magnetic action $\Theta_a(B)$ is a function of the magnetic field and is defined as

$$\Theta_a(B) = B \int_a \mathbf{A}(\mathbf{q}) \cdot d\mathbf{q} = B \int g_a(t) dt, \quad g_a(t) = \mathbf{A}(\mathbf{q}_a) \cdot \frac{d\mathbf{q}_a}{dt}, \quad (8)$$

where $\mathbf{A}(\mathbf{q})$ is the gauge potential which generates the unit magnetic field and $\mathbf{q}_a(t)$ describes a classical motion in the configuration space along the orbit a . Since the classical dynamics is chaotic, successive changes of the velocity can be regarded as independent events. Hence, if the time T elapsed on an orbit a is sufficiently large, the statistics of $g_a(t)$ can be regarded as that of the Gaussian white noise satisfying the correlation $\langle\langle g_a(t)g_a(t') \rangle\rangle = 2D\delta(t-t')$. The Gaussian average $\langle\langle \cdot \cdot \rangle\rangle$ is computed as a functional integral

$$\langle\langle F[g_a] \rangle\rangle = \frac{\int \mathcal{D}g_a \exp \left[-\frac{1}{4D} \int_0^T dt \{g_a(t)\}^2 \right] F[g_a]}{\int \mathcal{D}g_a \exp \left[-\frac{1}{4D} \int_0^T dt \{g_a(t)\}^2 \right]}. \quad (9)$$

It is known that this averaging yields the non-oscillatory terms of the spectral correlation [10, 11].

2.2. Pseudo-Orbit Representation of the Generating Function

Utilizing Eq.(6), we can rewrite $\det(E^+ - H)$ as

$$\begin{aligned} \det(E^+ - H) &\propto \exp \left\{ \int^{E^+} dE' \operatorname{Tr} \left(\frac{1}{E' - H} \right) \right\} \\ &\propto \exp \left[-i\pi \bar{N}(E^+) - \sum_a F_a e^{\frac{i}{\hbar} \{S_a(E^+) + \Theta_a(B)\}} \right]. \end{aligned} \quad (10)$$

Expanding the exponential we obtain an equivalent expression in terms of the pseudo orbits [16]:

$$\begin{aligned} \det(E^+ - H) &\propto e^{-i\pi\bar{N}(E^+)} \left\{ 1 - \sum_a F_a e^{\frac{i}{\hbar}(S_a + \Theta_a)} \right. \\ &\quad \left. + \frac{1}{2!} \sum_{a,a'} F_a F_{a'} e^{\frac{i}{\hbar}(S_a + \Theta_a)} e^{\frac{i}{\hbar}(S_{a'} + \Theta_{a'})} - \dots \right\} \\ &= e^{-i\pi\bar{N}(E^+)} \sum_A F_A (-1)^{N_A} e^{\frac{i}{\hbar}\{S_A(E^+) + \Theta_A(B)\}}, \end{aligned} \quad (11)$$

where A is the index of a pseudo orbit, which consists of component orbits, and N_A is the number of the component orbits. The functions S_A and Θ_A are the sums of the mechanical and magnetic actions of the component orbits, respectively, and F_A is the product of the stability amplitudes. The corresponding factor $\det(E^- - H)$ is given by the complex conjugation of $\det(E^+ - H)$ as

$$\det(E^- - H) \propto e^{i\pi\bar{N}(E^-)} \sum_A F_A^* (-1)^{N_A} e^{-\frac{i}{\hbar}\{S_A(E^-) + \Theta_A(B)\}}, \quad (12)$$

where an asterisk means a complex conjugate.

Berry and Keating [16] postulated that Eq.(11) and Eq.(12) become identical in the limit $E^+, E^- \rightarrow E$ and found a duality relation between the contribution of the pseudo-orbits with a duration larger than a half of the Heisenberg time $T_H = 2\pi\hbar\bar{\rho}$ and those with a duration shorter than $T_H/2$. Thus the sum over the long orbits can be treated as the complex conjugate of the sum over the short orbits. This leads to the so called Riemann-Siegel lookalike formula

$$\det(E - H) \propto e^{-i\pi\bar{N}(E)} \sum_{T_A < T_H/2} F_A (-1)^{N_A} e^{\frac{i}{\hbar}\{S_A(E) + \Theta_A(B)\}} + c.c., \quad (13)$$

This formula has an analogy in the corresponding expression of Riemann's zeta function.

Using Eqs.(10) and (11), we write the determinants of the inverse matrices as

$$\begin{aligned} \det(E^+ - H)^{-1} &\propto \exp \left[i\pi\bar{N}(E^+) + \sum_a F_a e^{\frac{i}{\hbar}\{S_a(E^+) + \Theta_a(B)\}} \right] \\ &= e^{i\pi\bar{N}(E^+)} \sum_A F_A e^{\frac{i}{\hbar}\{S_A(E^+) + \Theta_A(B)\}}, \end{aligned} \quad (14)$$

$$\begin{aligned} \det(E^- - H)^{-1} &\propto \exp \left[-i\pi\bar{N}(E^-) + \sum_a F_a^* e^{-\frac{i}{\hbar}\{S_a(E^-) + \Theta_a(B)\}} \right] \\ &= e^{-i\pi\bar{N}(E^-)} \sum_A F_A^* e^{-\frac{i}{\hbar}\{S_A(E^-) + \Theta_A(B)\}}. \end{aligned} \quad (15)$$

Inserting these expressions directly into the definition of the generating function, we obtain a semiclassical expression :

$$Z = \left\langle \left[e^{-i\pi\bar{N}(E_C)} \sum_{T_C < T_H/2} F_C (-1)^{N_C} e^{\frac{i}{\hbar}\{S_C(E_C) + \Theta_C\}} + c.c. \right] \right\rangle$$

$$\begin{aligned}
 & \times \left[e^{i\pi\bar{N}(E_D)} \sum_{T_D < T_H/2} F_D^*(-1)^{N_D} e^{-\frac{i}{\hbar}\{S_D(E_D)+\Theta_D\}} + c.c. \right] \\
 & \times e^{i\pi\bar{N}(E_A^+)} \sum_A F_A e^{\frac{i}{\hbar}\{S_A(E_A^+)+\Theta_A\}} \\
 & \times e^{-i\pi\bar{N}(E_B^-)} \sum_B F_B^* e^{-\frac{i}{\hbar}\{S_B(E_B^-)+\Theta_B\}} \Bigg\rangle. \tag{16}
 \end{aligned}$$

We used the Riemann-Siegel lookalike formula (13) and its complex conjugate for the numerator and the expressions (14)-(15) for the denominator. Because the expression (13) has two parts, Eq.(16) in total has four parts, each of which has the factor $e^{\pm i\pi[\bar{N}\{E+\epsilon_A/(2\pi\bar{\rho})\}-\bar{N}\{E-\epsilon_B/(2\pi\bar{\rho})\}]}$ or $e^{\pm i\pi[\bar{N}\{E+\epsilon_A/(2\pi\bar{\rho})\}+\bar{N}\{E-\epsilon_B/(2\pi\bar{\rho})\}]}$. However each part with the latter factor has no contribution because it is expected to vanish after averaging over E due to a rapid oscillation.

Eventually we obtain two parts for Z [17] as

$$Z = Z^{(1)} + Z^{(2)}, \tag{17}$$

where the first term $Z^{(1)}$ is written as

$$\begin{aligned}
 Z^{(1)} = & \sum_{\substack{A,B,C,D \\ (T_C, T_D < T_H/2)}} e^{i(\epsilon_A^+ - \epsilon_B^- - \epsilon_C + \epsilon_D)/2} \\
 & \times \left\langle F_A F_B^* F_C F_D^* (-1)^{N_C + N_D} e^{\frac{i}{\hbar}(S_A(E) - S_B(E) + S_C(E) - S_D(E))} \right\rangle \\
 & \times \langle \langle e^{\frac{i}{\hbar}(\Theta_A - \Theta_B + \Theta_C - \Theta_D)} \rangle \rangle e^{\frac{i}{T_H}(T_A \epsilon_A^+ - T_B \epsilon_B^- + T_C \epsilon_C - T_D \epsilon_D)}. \tag{18}
 \end{aligned}$$

Here the symbol $\langle \langle \dots \rangle \rangle$ implies the average over the weight Eq.(9). To derive Eq.(18), we used the expansion $\bar{N}(E \pm \epsilon/2\pi\bar{\rho}) \sim \bar{N}(E) \pm \epsilon/2\pi$, and $S(E \pm \epsilon/2\pi\bar{\rho})/\hbar \sim S(E)/\hbar \pm T\epsilon/T_H$. The term $Z^{(2)}$ is obtained from $Z^{(1)}$ by exchanging the variables ϵ_C and ϵ_D as

$$Z^{(2)}(\epsilon_A, \epsilon_B, \epsilon_C, \epsilon_D) = Z^{(1)}(\epsilon_A, \epsilon_B, \epsilon_D, \epsilon_C). \tag{19}$$

Utilizing these expressions, we write the second derivatives of Z , which gives the spectral correlation $C(\epsilon)$, as

$$\left. \frac{\partial^2 Z}{\partial \epsilon_A \partial \epsilon_B} \right|_{||} = \left. \frac{\partial^2 Z^{(1)}}{\partial \epsilon_A \partial \epsilon_B} \right|_{||} + \left. \frac{\partial^2 Z^{(2)}}{\partial \epsilon_A \partial \epsilon_B} \right|_{||} = \left. \frac{\partial^2 Z^{(1)}}{\partial \epsilon_A \partial \epsilon_B} \right|_{||} + \left. \frac{\partial^2 Z^{(1)}}{\partial \epsilon_A \partial \epsilon_B} \right|_{\times}, \tag{20}$$

where \times denotes the identification $\epsilon_A = -\epsilon_B = \epsilon^+$, $\gamma \rightarrow +0$; $\epsilon_D = -\epsilon_C = \epsilon$. In Ref. [14], the identifications with the symbols $||$ and \times were called ‘‘column-wise’’ and ‘‘crosswise’’, respectively.

3. Consecutive Approximation Orders in Terms of Periodic Orbits

3.1. Diagonal Approximation

Let us first consider the diagonal approximation, in which different orbits are assumed to be uncorrelated, except for the orbits obtained from each other by time reversal. We

express Eq.(18) in terms of the periodic orbits, not using pseudo orbits. On account of Eq.(10), the generating function $Z^{(1)}$ can be written in terms of the periodic orbits:

$$\begin{aligned} Z^{(1)} &= e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \left\langle \exp \left[\sum_a e^{\frac{i}{\hbar}\{S_a(E) + \Theta_a\}} f_{AC}^a + e^{-\frac{i}{\hbar}\{S_a(E) + \Theta_a\}} f_{BD}^{a*} \right] \right\rangle, \\ f_{AC}^a &= F_a(E) \left(e^{\frac{iT_a}{T_H}\epsilon_A} - e^{\frac{iT_a}{T_H}\epsilon_C} \right), \\ f_{BD}^{a*} &= F_a^*(E) \left(e^{-\frac{iT_a}{T_H}\epsilon_B} - e^{-\frac{iT_a}{T_H}\epsilon_D} \right). \end{aligned} \quad (21)$$

It can be rewritten as an averaged product over the *pairs* (a, a^{TR}) of the mutually time reversed orbits; considering that a time-reversed orbit has a different sign in the gauge-potential term, we obtain

$$\begin{aligned} Z^{(1)} &= e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \left\langle \prod_{(a, a^{TR})} \exp \left[e^{\frac{i}{\hbar}S_a} f_{AC}^a (e^{\frac{i}{\hbar}\Theta_a} + e^{-\frac{i}{\hbar}\Theta_a}) \right. \right. \\ &\quad \left. \left. + e^{-\frac{i}{\hbar}S_a} f_{BD}^{a*} (e^{\frac{i}{\hbar}\Theta_a} + e^{-\frac{i}{\hbar}\Theta_a}) \right] \right\rangle. \end{aligned} \quad (22)$$

Now let us calculate the average. As the factors in the product are assumed to be uncorrelated, the averaged product of the factors can be replaced by the product of the averaged factors. The average over the energy E is replaced by an average

$$\langle \dots \rangle = \frac{1}{2\pi\hbar} \int_0^{2\pi\hbar} (\dots) dS_a \quad (23)$$

over the classical action S_a . The terms containing the magnetic action are averaged over the Gaussian white noise (see Eq.(9) as

$$\langle \langle e^{\frac{2i\Theta_a}{\hbar}} \rangle \rangle = e^{-bT/T_H}, \quad (24)$$

where $b = 4B^2DT_H/\hbar^2$. Considering that the coefficients f_{AC}^a, f_{BD}^a are exponentially small for the relevant long orbits, we expand each factor up to the second order and doubly average the result. Then we find

$$\begin{aligned} &\left\langle \exp \left[e^{\frac{i}{\hbar}S_a} f_{AC}^a (e^{\frac{i}{\hbar}\Theta_a} + e^{-\frac{i}{\hbar}\Theta_a}) + e^{-\frac{i}{\hbar}S_a} f_{BD}^{a*} (e^{\frac{i}{\hbar}\Theta_a} + e^{-\frac{i}{\hbar}\Theta_a}) \right] \right\rangle \\ &\approx 1 + f_{AC}^a f_{BD}^{a*} \left(2 + \langle \langle e^{\frac{i2\Theta}{\hbar}} \rangle \rangle + \langle \langle e^{-\frac{i2\Theta}{\hbar}} \rangle \rangle \right) \\ &= 1 + 2f_{AC}^a f_{BD}^{a*} (1 + e^{-bT_a/T_H}) \\ &\approx \exp [2f_{AC}^a f_{BD}^{a*} (1 + e^{-bT_a/T_H})] \end{aligned} \quad (25)$$

It is now straightforward to obtain the expression

$$\begin{aligned} Z_{\text{diag}}^{(1)} &\sim e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \\ &\times \exp \left[\sum_a |F_a(E)|^2 \left(e^{\frac{iT_a}{T_H}\epsilon_A} - e^{\frac{iT_a}{T_H}\epsilon_C} \right) \left(e^{-\frac{iT_a}{T_H}\epsilon_B} - e^{-\frac{iT_a}{T_H}\epsilon_D} \right) (1 + e^{-bT_a/T_H}) \right], \end{aligned} \quad (26)$$

where summation in the exponent is carried over orbits, not orbit pairs; for compensation, the factor 2 is dropped.

We can further proceed with the calculation by using Hannay and Ozorio de Almeida (HOdA) sum rule [18]

$$\sum_a |F_a|^2(\cdot) \sim \int_{T_0}^{\infty} \frac{dT}{T}(\cdot), \quad (27)$$

where the lower limit of the integration is the minimum period T_0 ; in the semiclassical limit it can be replaced by zero. Then we immediately obtain the result of the diagonal approximation:

$$\begin{aligned} Z_{\text{diag}}^{(1)} &= e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \\ &\times \left\{ \frac{(\epsilon_C - \epsilon_B)(\epsilon_A - \epsilon_D)}{(\epsilon_A - \epsilon_B)(\epsilon_C - \epsilon_D)} \right\} \left\{ \frac{(\epsilon_C - \epsilon_B + ib)(\epsilon_A - \epsilon_D + ib)}{(\epsilon_A - \epsilon_B + ib)(\epsilon_C - \epsilon_D + ib)} \right\}. \end{aligned} \quad (28)$$

3.2. Second Order Contribution

Let us now go beyond the diagonal approximation. In order to calculate the off-diagonal contributions, we can utilize the method developed in [3, 14]. In that method, the periodic orbits are divided into links and encounters. An encounter is a part of an orbit where different links meet together, and orbit stretches connect the links inside the encounter region. It is assumed that the contributions come from the periodic-orbit pairs with the same (or time-reversed) links which are differently connected inside the encounter regions. Due to the different connections in the encounter regions, there is a small but significant difference in actions between the original and partner orbits. The gauge potential acts on both links and encounters. The gauge potential constructively interferes when the original and partner orbits go in the opposite directions. On the other hand, when the directions are the same, a destructive interference takes place.

The off-diagonal contribution is formally written by using the pseudo-orbit representation (18) as

$$\begin{aligned} Z_{\text{off}}^{(1)} &= \sum'_{A,B,C,D} \left\langle F_A F_B^* F_C F_D^* (-1)^{N_C + N_D} e^{\frac{i}{\hbar} \Delta S} \right\rangle \langle \langle e^{\frac{i}{\hbar} \Delta \Theta} \rangle \rangle \\ &\times e^{\frac{i}{T_H} (T_A \epsilon_A - T_B \epsilon_B + T_C \epsilon_C - T_D \epsilon_D)}, \end{aligned} \quad (29)$$

where $\sum'_{A,B,C,D}$ means a summation over the pseudo orbits forming links and encounters. ΔS represents a sum of action differences in the encounter regions, and $\Delta \Theta$ expresses the effect of interferences of the gauge potential. The overall sum $Z^{(1)}$ is obtained by multiplying the two contributions

$$Z^{(1)} = Z_{\text{diag}}^{(1)} Z_{\text{off}}^{(1)}. \quad (30)$$

Let us now calculate the leading (second order) term of the off-diagonal contribution. We first discuss the Sieber-Richter (SR) pairs, one of which is depicted in Figure 1. A SR pair has one encounter and two links, and the original and partner orbits go in the opposite directions in one of the links. In Figure 1, the solid curve is the original orbit and the dashed curve is the partner. A pseudo-orbit index “A” or “C” is assigned to the original orbit, and “B” or “D” is to the partner. There are in total

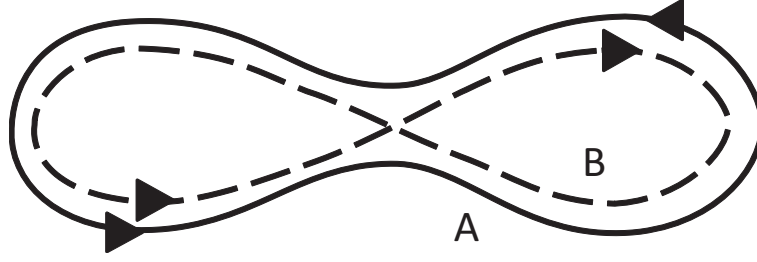


Figure 1. An example of the Sieber-Richter (SR) pairs, where “A” is assigned to the original orbit (the solid curve) and “B” is to the partner (the dashed curve).

A	→	→	→	→	←	←	←	←	∅	∅	∅	∅	∅	∅	∅	∅
B	- - →	← - -	∅	∅	- - →	← - -	∅	∅	- - →	← - -	∅	∅	- - →	← - -	∅	∅
C	∅	∅	∅	∅	∅	∅	∅	∅	→	→	→	→	←	←	←	←
D	∅	∅	- - →	← - -	∅	∅	- - →	← - -	∅	∅	- - →	← - -	∅	∅	- - →	← - -

Table 1. A schematic table for the 16 possibilities of the SR pairs. Here \emptyset means the absence in the diagram.

16 possibilities, as listed in Table 1. On the Poincaré section within the encounter region, the phase space can be furnished with canonical coordinates u and s , which are the coordinates in the unstable and stable directions, respectively. Let us suppose that the original orbit pierces the Poincaré section at the origin $(0, 0)$ and at the point (u, s) . To demand that the piercing points in the encounter are mutually close, we introduce a bound c and assume that $|u|, |s| < c$. Then we can estimate the duration t_{enc} of the encounter and the action difference ΔS of the original and partner orbits as $t_{\text{enc}} = \frac{1}{\lambda} \ln \frac{c^2}{|us|}$ and $\Delta S = us$, where λ is the Lyapunov exponent. Now we need to estimate the number of encounters in one periodic orbit of a period T . This can be computed as

$$\int_{-c}^c du ds \int_0^{T-2t_{\text{enc}}} dt \frac{T}{2^2 t_{\text{enc}} \Omega}, \quad (31)$$

where $\Omega^{-1} \left(= \frac{1}{2\pi\hbar T_H} \right)$ is the ergodic return probability per unit action. The combinatorial factor 2^2 implies that one of the two encounter stretches is chosen as the first stretch and one of the two directions is chosen as the positive direction. The factor $T \int_0^{T-2t_{\text{enc}}} dt$ indicates that one of the two piercings occurs in the time interval $[0, T]$ and that the time t elapsed on one of the links lies in $[0, T-2t_{\text{enc}}]$. The contribution from the gauge potential has to be taken into account on one of the links, where the original and partner orbits go in the opposite directions. In the SR pair, a constructive interference of the gauge potential does not exist in the encounter region. Thus the contribution from the terms depicted in Figure 1 can be calculated as

$$Z_{\text{Fig.1}}^{(1)} = \sum_a |F_a|^2 \int ds du \int_0^{T-2t_{\text{enc}}} dt \frac{T}{2^2 t_{\text{enc}} \Omega} e^{isu/\hbar} e^{-bt/T_H} e^{i(\epsilon_A - \epsilon_B)T/T_H}. \quad (32)$$

The HOdA sum rule [18] changes the sum $\sum_a |F_a|^2 T_a$ into $\int dT$. Hence, Eq.(32) is rewritten as

$$\frac{1}{4} \int ds du \int_{2t_{\text{enc}}}^{\infty} dT \int_0^{T-2t_{\text{enc}}} dt \frac{1}{\Omega t_{\text{enc}}} e^{isu/\hbar} e^{-bt/T_H} e^{i(\epsilon_A - \epsilon_B)T/T_H}. \quad (33)$$

To clarify the contributions from the encounter and links, we introduce new variables as

$$T = t_1 + t_2 + 2t_{\text{enc}}, \quad (34)$$

$$t = t_2. \quad (35)$$

Then the expression is simplified as

$$\begin{aligned} & \frac{1}{4} \left\{ \frac{1}{T_H} \int_0^{\infty} dt_1 e^{i(\epsilon_A - \epsilon_B)t_1/T_H} \right\} \left\{ \frac{1}{T_H} \int_0^{\infty} dt_2 e^{i(\epsilon_A - \epsilon_B)t_2/T_H - bt_2/T_H} \right\} \\ & \times T_H^2 \int ds \int du \frac{1}{\Omega t_{\text{enc}}} e^{isu/\hbar} e^{i(\epsilon_A - \epsilon_B)2t_{\text{enc}}/T_H}. \end{aligned} \quad (36)$$

Now we can identify the contributions from the encounter and links: in the above expression, the integrals over t_1 and t_2 are interpreted as the link-contributions, and the integral over (u, s) is the encounter-contribution. We expand the expression in t_{enc} and extract the constant term, which is expected to survive in the semiclassical limit $\hbar \rightarrow 0$. Then we obtain

$$\frac{1}{2i(\epsilon_A - \epsilon_B + ib)}. \quad (37)$$

We repeat similar calculations for the possibilities listed in Table 1. Summing up the results, we find the contribution from the SR pairs

$$\begin{aligned} Z_{\text{SR}}^{(1)} &= \frac{2}{i(\epsilon_A - \epsilon_B + ib)} - \frac{2}{i(\epsilon_A - \epsilon_D + ib)} \\ &\quad - \frac{2}{i(\epsilon_C - \epsilon_B + ib)} + \frac{2}{i(\epsilon_C - \epsilon_D + ib)}. \end{aligned} \quad (38)$$

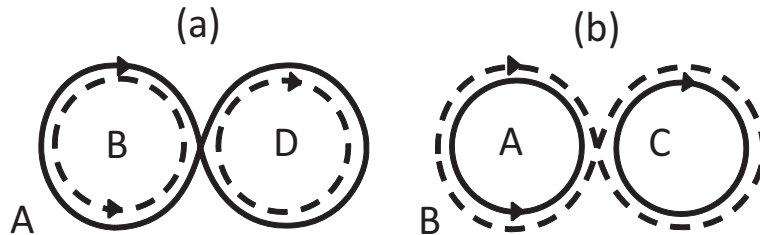


Figure 2. Examples of the parallel Sieber-Richter (aSR) pairs.

In the evaluation of the second order term, we need to consider the other types of pseudo-orbit pairs, two examples of which are shown in Figure 2. These are called “parallel Sieber-Richter” (aSR) pairs, following Ref. [14]. In Figure 2(a), an index “A” or “C” is assigned to the original orbit (solid curve), and “B” and “D” are to the

components of the partner pseudo orbit. In Figure 2(b), on the other hand, “A” and “C” are assigned to the components of the original pseudo orbit, while “B” or “D” is to the partner orbit. Let us calculate the contribution from Figure 2(a). In both of the two links, the original orbit and the partner pseudo orbit go in the opposite directions. In the encounter, the stretches of the original orbit and those of the partner pseudo orbit also go in the opposite directions. Hence, in this case there are constructive interferences of the gauge potential in all parts of the pair. The link contribution is given by

$$\frac{1}{T_H} \int_0^\infty dt_1 e^{i(\epsilon_A - \epsilon_B)t_1/T_H - bt_1/T_H} \times \frac{1}{T_H} \int_0^\infty dt_2 e^{i(\epsilon_A - \epsilon_D)t_2/T_H - bt_2/T_H} \quad (39)$$

and the encounter yields

$$T_H^2 \int du \int ds \frac{1}{\Omega t_{\text{enc}}} e^{isu/\hbar} e^{i\{(2\epsilon_A - \epsilon_B - \epsilon_D)/T_H + i4b/T_H\}t_{\text{enc}}}. \quad (40)$$

In general, when the original (partner) pseudo orbit passes the encounter n_+ and n_- times (n'_+ and n'_- times) in the positive and negative directions, respectively, an exponential factor $\exp(in_{\text{enc}}^2 b t_{\text{enc}}/T_H)$ with

$$n_{\text{enc}} = \frac{1}{2} |n_+ - n_- - n'_+ + n'_-| \quad (41)$$

appears in the encounter contribution [10]. This is a consequence of the stochastic behavior of the gauge potential (9). Extracting the term surviving in the semiclassical limit $\hbar \rightarrow \infty$, we obtain the contribution from Figure 2(a) as

$$\begin{aligned} Z_{\text{Fig.2(a)}}^{(1)} &= \frac{1}{4} \left\{ \frac{1}{T_H} \int_0^\infty dt_1 e^{i(\epsilon_A - \epsilon_B)t_1/T_H - bt_1/T_H} \right\} \\ &\times \left\{ \frac{1}{T_H} \int_0^\infty dt_2 e^{i(\epsilon_A - \epsilon_D)t_2/T_H - bt_2/T_H} \right\} \\ &\times T_H^2 \int du \int ds \frac{1}{\Omega t_{\text{enc}}} e^{isu/\hbar} e^{i\{(2\epsilon_A - \epsilon_B - \epsilon_D)/T_H + i4b/T_H\}t_{\text{enc}}} \\ &= \frac{-i}{4} \frac{2\epsilon_A - \epsilon_B - \epsilon_D + i4b}{(\epsilon_A - \epsilon_B + ib)(\epsilon_A - \epsilon_D + ib)} \end{aligned} \quad (42)$$

A calculation for the diagram in Figure 2(b) is similar and yields

$$\begin{aligned} Z_{\text{Fig.2(b)}}^{(1)} &= \frac{1}{4} \left\{ \frac{1}{T_H} \int_0^\infty dt_1 e^{i(\epsilon_A - \epsilon_B)t_1/T_H - bt_1/T_H} \right\} \\ &\times \left\{ \frac{1}{T_H} \int_0^\infty dt_2 e^{i(\epsilon_C - \epsilon_B)t_2/T_H - bt_2/T_H} \right\} \\ &\times T_H^2 \int du \int ds \frac{1}{\Omega t_{\text{enc}}} e^{isu/\hbar} e^{i\{(\epsilon_A + \epsilon_C - \epsilon_B)/T_H + i4b/T_H\}t_{\text{enc}}} \\ &= \frac{-i}{4} \frac{\epsilon_A + \epsilon_C - \epsilon_B + i4b}{(\epsilon_A - \epsilon_B + ib)(\epsilon_C - \epsilon_B + ib)} \end{aligned} \quad (43)$$

In order to obtain the complete set of the aSR pairs, we need to change the assignment of the indices A, B, C and D as well as the directions of the pseudo-orbit components. There are 64 diagrams for each of the types in Figure 2(a) and Figure

2(b). Thus totally 128 diagrams have to be considered. Summing up the results by using "Mathematica", we find the total contribution from the aSR pairs

$$Z_{\text{aSR}}^{(1)} = 2b(\epsilon_A - \epsilon_C)(\epsilon_B - \epsilon_D) \times \left\{ \frac{1}{(\epsilon_C - \epsilon_D + ib)^2(\epsilon_A - \epsilon_D + ib)(\epsilon_C - \epsilon_B + ib)} - \frac{1}{(\epsilon_A - \epsilon_B + ib)^2(\epsilon_A - \epsilon_D + ib)(\epsilon_C - \epsilon_B + ib)} \right\}. \quad (44)$$

Note that, in both of the limits $b \rightarrow 0$ and $b \rightarrow \infty$, this expression vanishes: It is finite only in the crossover domain.

3.3. Higher Order Terms

Let us consider the higher order terms in the periodic-orbit expansion. Suppose that the pseudo-orbit pair includes V encounters, as schematically drawn in Figure 3. In the figure, the ports (the end points of the encounters) are depicted as black dots. A pseudo orbit is formed by connecting the ports by stretches within the encounters and by links out of them. In order to see the general structures of the higher order terms, we need to count the number of pseudo-orbit pairs, carefully avoiding overcountings.

As we have seen in the calculation of the SR and aSR pairs, we can identify the contribution from links and that from the encounters. That is, a link gives a contribution

$$\frac{i}{\epsilon_{\text{AorC}} - \epsilon_{\text{BorD}} + i\mu b}, \quad (45)$$

where $\mu = 0$ if the link is traversed with the same sense in the original and partner pseudo orbits, and $\mu = 1$ if the senses of traversal are opposite. On the other hand, the encounter has the contribution

$$i(m_A\epsilon_A - m_B\epsilon_B + m_C\epsilon_C - m_D\epsilon_D) - bn_{\text{enc}}^2, \quad (46)$$

where m_A, \dots, m_D are the number of the stretches belonging to A, B, C, D , respectively.

Now we estimate the number of over-countings, when there are V encounters. Obviously, the ordering of the encounters is arbitrary. Therefore there are $V!$ overcountings from the ordering of the encounters. Suppose that the j th encounter has ℓ_j stretches of the original pseudo orbit. Inside the j th encounter, we also overcount ℓ_j times, since it is arbitrary which port is chosen as the first. Since the stretches are no longer required to point from left to right, we can relabel the left side as the right side and vice versa. This requires one more overcounting factor 2^V .

Thus the periodic-orbit expansion of $Z_{\text{off}}^{(1)}$ is formally written as

$$Z_{\text{off}}^{(1)} = \sum_{V=1}^{\infty} \frac{1}{V!} \sum_{\ell_1, \ell_2, \dots, \ell_V} \sum_{\text{pseudo-orbit pairs}} (-1)^{n_C + n_D} \times \frac{\prod_{j=1}^V \left(\frac{1}{2\ell_j} \right) \{i(m_{A_j}\epsilon_A - m_{B_j}\epsilon_B + m_{C_j}\epsilon_C - m_{D_j}\epsilon_D) - bn_{\text{enc},j}^2\}}{\prod_{\text{links}} (-i) \{(\epsilon_{\text{AorC}} - \epsilon_{\text{BorD}}) + i\mu b\}}, \quad (47)$$

where the second sum is over all possible combinations of the number of stretches, and the third sum means counting over all the possible pseudo-orbit pairs when $(\ell_1, \ell_2, \dots, \ell_V)$ are given.

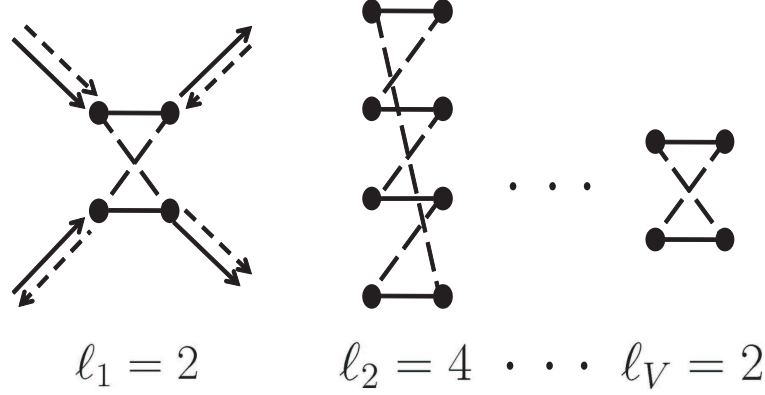


Figure 3. A schematic picture of a pseudo orbit pair with V encounters. The links are shown only for the first encounter. The solid lines depict the original pseudo orbit and the dashed lines depict the partner.

4. Nonlinear Sigma Model in GOE-GUE Crossover Domain

Let us examine the random matrix theory prediction for the generating function and derive an expansion which can directly be compared with the semiclassical formula. For that purpose, using the Bosonic variant of the replica trick we analyze the nonlinear sigma (NLS) model. Compared with the Fermionic replica and the supersymmetric version of the sigma model [19] the Bosonic replica has the disadvantage that it yields only the part $Z^{(1)}$ of the generating function. However we choose it because it is technically much simpler, and, anyhow, the complement $Z^{(2)}$ can be obtained from $Z^{(1)}$ by simply swapping arguments as $\epsilon_C \leftrightarrow \epsilon_D$. As shown in the following sections, the ensuing perturbative expansion of the NLS model is equivalent to the semiclassical periodic-orbit expansion for $Z^{(1)}$.

As derived in Appendix A, the NLS model for the generating function is written as

$$\begin{aligned}
 \mathcal{Z} &= \lim_{r \rightarrow 0} \int d[B] \exp\{\mathcal{S}(B)\}, \\
 \mathcal{S}(B) &= \frac{i}{2}(\epsilon_A - \epsilon_B + \epsilon_C - \epsilon_D) \\
 &\quad + \frac{i}{2}\text{Tr}(\tilde{\epsilon}_{AC}BB^\dagger) - \frac{i}{2}\text{Tr}(\tilde{\epsilon}_{BD}B^\dagger B) - 2s\text{Tr}(BB^\dagger) + 2s\text{Tr}(B\tau_3 B^\dagger \tau_3) \\
 &\quad + \frac{i}{2} \sum_{m=2}^{\infty} [\text{Tr}\{\tilde{\epsilon}_{AC}(BB^\dagger)^m\} - \text{Tr}\{\tilde{\epsilon}_{BD}(B^\dagger B)^m\}] - 2s \sum_{m=2}^{\infty} \text{Tr}(BB^\dagger)^m \\
 &\quad - s \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \text{Tr}\{(BB^\dagger)^m \tau_3 (BB^\dagger)^{m'} \tau_3\}
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 & -s \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \text{Tr} \left\{ (B^\dagger B)^m \tau_3 (B^\dagger B)^{m'} \tau_3 \right\} \\
 & + 2s \sum_{\substack{m, m'=0 \\ (m, m') \neq (0, 0)}}^{\infty} \text{Tr} \left\{ (BB^\dagger)^m B \tau_3 (B^\dagger B)^{m'} B^\dagger \tau_3 \right\}, \tag{49}
 \end{aligned}$$

where the measure in the integral is defined as $d[B] = \prod_{i,k} d\text{Re}B_{ik} d\text{Im}B_{ik}$. $\tilde{\epsilon}_{AC}$, $\tilde{\epsilon}_{BD}$ and τ_3 are $2r \times 2r$ matrices given by

$$\tilde{\epsilon}_{AC} = \begin{pmatrix} \hat{\epsilon}_{AC} & 0 \\ 0 & \hat{\epsilon}_{AC} \end{pmatrix}, \quad \hat{\epsilon}_{AC} = \text{diag}(\epsilon_A, \overbrace{\epsilon_C, \dots, \epsilon_C}^{r-1}), \tag{50}$$

$$\tilde{\epsilon}_{BD} = \begin{pmatrix} \hat{\epsilon}_{BD} & 0 \\ 0 & \hat{\epsilon}_{BD} \end{pmatrix}, \quad \hat{\epsilon}_{BD} = \text{diag}(\epsilon_B, \overbrace{\epsilon_D, \dots, \epsilon_D}^{r-1}), \tag{51}$$

$$\tau_3 = \text{diag}(\overbrace{1, 1, \dots, 1}^r, \overbrace{-1, -1, \dots, -1}^r). \tag{52}$$

The integrations are performed over all the matrix elements of B with the range $(-\infty, \infty)$. The variable r is the number of replica spaces, and we finally take the limit $r \rightarrow 0$.

4.1. Diagonal Term

Let us calculate the diagonal term $\mathcal{Z}_{\text{diag}}$ of the generating function. The diagonal contribution comes from the quadratic terms of $\mathcal{S}(B)$ as

$$\begin{aligned}
 \mathcal{Z}_{\text{diag}} = & e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \lim_{r \rightarrow 0} \int d[B] \exp \left\{ \frac{i}{2} \text{Tr}(\tilde{\epsilon}_{AC} B B^\dagger) - \frac{i}{2} \text{Tr}(\tilde{\epsilon}_{BD} B^\dagger B) \right. \\
 & \left. - 2s \text{Tr}(B B^\dagger) + 2s \text{Tr}(B \tau_3 B \tau_3) \right\}. \tag{53}
 \end{aligned}$$

In order to explicitly calculate this integral, we decompose the B matrix as

$$B = \begin{pmatrix} B^{++} & B^{+-} \\ B^{-+} & B^{--} \end{pmatrix}, \tag{54}$$

where each matrix $B^{\alpha\beta}$ is an $r \times r$ matrix. We note that the symmetry (A.33) requires the relations

$$(B^{++})^\dagger = -(B^{--})^T, \tag{55}$$

$$(B^{-+})^\dagger = -(B^{+-})^T. \tag{56}$$

Using these relations, we calculate (53) as

$$\begin{aligned}
 \mathcal{Z}_{\text{diag}} = & e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \\
 & \times \lim_{r \rightarrow 0} \int d[B] \exp \left[\sum_{\alpha, \beta = \pm} \sum_{k, \ell=1}^r \left\{ \frac{i}{2} (\hat{\epsilon}_{AC, k} - \hat{\epsilon}_{BD, \ell}) - 2s + 2s\alpha\beta \right\} |B_{k, \ell}^{\alpha, \beta}|^2 \right] \\
 = & e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \\
 & \times \lim_{r \rightarrow 0} \frac{-\pi}{i(\epsilon_A - \epsilon_B) - 8s} \frac{-\pi}{i(\epsilon_A - \epsilon_B)} \left\{ \frac{-\pi}{i(\epsilon_A - \epsilon_D) - 8s} \frac{-\pi}{i(\epsilon_A - \epsilon_D)} \right\}^{r-1}
 \end{aligned}$$

$$\times \left\{ \frac{-\pi}{i(\epsilon_C - \epsilon_B) - 8s} \frac{-\pi}{i(\epsilon_C - \epsilon_B)} \right\}^{r-1} \left\{ \frac{-\pi}{i(\epsilon_C - \epsilon_D) - 8s} \frac{-\pi}{i(\epsilon_C - \epsilon_D)} \right\}^{(r-1)(r-1)}. \quad (57)$$

Taking the limit $r \rightarrow 0$, we obtain

$$\begin{aligned} \mathcal{Z}_{\text{diag}} &= e^{i(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D)/2} \\ &\times \left\{ \frac{(\epsilon_A - \epsilon_D + i8s)(\epsilon_C - \epsilon_B + i8s)}{(\epsilon_A - \epsilon_B + i8s)(\epsilon_C - \epsilon_D + i8s)} \right\} \left\{ \frac{(\epsilon_A - \epsilon_D)(\epsilon_C - \epsilon_B)}{(\epsilon_A - \epsilon_B)(\epsilon_C - \epsilon_D)} \right\}. \end{aligned} \quad (58)$$

This is identical to the semiclassical formula, if $8s$ is replaced by b .

4.2. Expansion of the Generating Function

The generating function \mathcal{Z} is formally expanded as

$$\mathcal{Z} = \mathcal{Z}_{\text{diag}}(1 + \mathcal{Z}_{\text{off}}), \quad (59)$$

$$\begin{aligned} \mathcal{Z}_{\text{off}} &= \lim_{r \rightarrow 0} \left\langle \exp \left(\sum_{\ell=2}^{\infty} \mathcal{F}_{\ell} \right) \right\rangle - 1 \\ &= \lim_{r \rightarrow 0} \left\{ \sum_{\ell} \langle \mathcal{F}_{\ell} \rangle + \frac{1}{2!} \sum_{\ell, \ell'} \langle \mathcal{F}_{\ell} \mathcal{F}_{\ell'} \rangle + \dots \right\}, \end{aligned} \quad (60)$$

where \mathcal{F}_{ℓ} is written as

$$\begin{aligned} \mathcal{F}_{\ell} &= \frac{i}{2} \text{Tr} \{ \tilde{\epsilon}_{AC} (BB^{\dagger})^{\ell} \} - \frac{i}{2} \text{Tr} \{ \tilde{\epsilon}_{BD} (B^{\dagger}B)^{\ell} \} - 2s \text{Tr} (BB^{\dagger})^{\ell} \\ &- s \sum_{m=1}^{\ell-1} \text{Tr} \{ (BB^{\dagger})^m \tau_3 (BB^{\dagger})^{\ell-m} \tau_3 \} - s \sum_{m=1}^{\ell-1} \text{Tr} \{ (B^{\dagger}B)^m \tau_3 (B^{\dagger}B)^{\ell-m} \tau_3 \} \\ &+ 2s \sum_{m=0}^{\ell-1} \text{Tr} \{ (BB^{\dagger})^m B \tau_3 (B^{\dagger}B)^{\ell-m-1} B^{\dagger} \tau_3 \}. \end{aligned} \quad (61)$$

The symbol $\langle \dots \rangle$ in Eq.(60) means an average computed over the Gaussian weight in (53) with a normalization factor. Perturbative expansions will be derived by means of the Wick theorem on this Gaussian average. Thus the Gaussian average plays a crucial role in establishing the relationship between the periodic-orbit theory and random matrices. The Gaussian averages of the two-point correlation functions of B are evaluated as

$$\left\langle (B_{k,\ell}^{\alpha\beta})^* B_{k',\ell'}^{\alpha'\beta'} \right\rangle = - \frac{\delta_{k,k'} \delta_{\ell,\ell'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}}{i(\hat{\epsilon}_{AC,k} - \hat{\epsilon}_{BD,\ell}) - 4s|\alpha - \beta|}, \quad (62)$$

$$\left\langle B_{k,\ell}^{\alpha\beta} B_{k',\ell'}^{\alpha'\beta'} \right\rangle = \frac{\delta_{k,k'} \delta_{\ell,\ell'} \delta_{\alpha,\bar{\alpha}'} \delta_{\beta,\bar{\beta}'}}{i(\hat{\epsilon}_{AC,k} - \hat{\epsilon}_{BD,\ell}) - 4s|\alpha - \beta|}, \quad (63)$$

where $\bar{\alpha}'$ and $\bar{\beta}'$ denote the opposites of the signs α' and β' , respectively, and k, k', ℓ and ℓ' take the values $1, \dots, r$.

In order to obtain a simple expression for \mathcal{F}_{ℓ} in terms of the matrix elements of B , we make use of the cyclic property of a trace, i.e., $\text{Tr}(\mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_L) = \text{Tr}(\mathcal{O}_2 \mathcal{O}_3 \dots \mathcal{O}_1) = \dots = \text{Tr}(\mathcal{O}_L \mathcal{O}_1 \dots \mathcal{O}_{L-1})$. Due to this cyclic property, ℓ equivalent but seemingly

different expressions for \mathcal{F}_ℓ are obtained. Taking the average of these expressions, we find the following compact formula

$$\begin{aligned} \mathcal{F}_\ell = & \frac{1}{\ell} \sum_{\alpha_1, \dots, \alpha_{2\ell}} \sum_{k_1, \dots, k_{2\ell}} B_{k_1, k_2}^{\alpha_1, \alpha_2} (B_{k_3, k_2}^{\alpha_3, \alpha_2})^* \dots B_{k_{2\ell-1}, k_{2\ell-1}}^{\alpha_{2\ell-1}, \alpha_{2\ell}} (B_{k_1, k_{2\ell}}^{\alpha_1, \alpha_{2\ell}})^* \\ & \times \frac{1}{2} \left[\sum_{j=1}^{\ell} i(\hat{\epsilon}_{AC, k_{2j-1}} - \hat{\epsilon}_{BD, k_{2j}}) - 8s \left\{ \frac{1}{2} \sum_{j=1}^{\ell} (\alpha_{2j} - \alpha_{2j-1}) \right\}^2 \right]. \end{aligned} \quad (64)$$

As clarified below, the squared factor corresponds to the factor n_{enc}^2 in the semiclassical expression [10]. In order to evaluate the expansion (60), general correlation functions of B are necessary. We are able to calculate them by using the Wick theorem. In the next subsection, we obtain the second order term which yields the leading off-diagonal contribution.

4.2.1. Second Order Contribution Second order contribution in \mathcal{Z}_{off} results from the quartic terms in B . The quartic terms are simply given by

$$\begin{aligned} \langle \mathcal{F}_2 \rangle = & \sum_{\alpha_1, \dots, \alpha_4 = \pm} \sum_{k_1, \dots, k_4=1}^r \left(\frac{1}{2} \right)^2 \\ & \times \left\{ i[\hat{\epsilon}_{AC, k_1} + \hat{\epsilon}_{AC, k_3} - (\hat{\epsilon}_{BD, k_2} + \hat{\epsilon}_{BD, k_4})] - 8s \left(\frac{\alpha_1 + \alpha_3}{2} - \frac{\alpha_2 + \alpha_4}{2} \right)^2 \right\} \\ & \times \left\langle B_{k_1, k_2}^{\alpha_1, \alpha_2} (B_{k_3, k_2}^{\alpha_3, \alpha_2})^* B_{k_3, k_4}^{\alpha_3, \alpha_4} (B_{k_1, k_4}^{\alpha_1, \alpha_4})^* \right\rangle. \end{aligned} \quad (65)$$

We employ the Wick theorem to write this in terms of the two-point correlation functions (62) and (63). That is, a four-point correlation function is expanded as

$$\begin{aligned} \left\langle B_{k_1, k_2}^{\alpha_1, \alpha_2} (B_{k_3, k_2}^{\alpha_3, \alpha_2})^* B_{k_3, k_4}^{\alpha_3, \alpha_4} (B_{k_1, k_4}^{\alpha_1, \alpha_4})^* \right\rangle = & \left\langle B_{k_1, k_2}^{\alpha_1, \alpha_2} B_{k_3, k_4}^{\alpha_3, \alpha_4} \right\rangle \left\langle (B_{k_3, k_2}^{\alpha_3, \alpha_2})^* (B_{k_1, k_4}^{\alpha_1, \alpha_4})^* \right\rangle \\ & + \left\langle B_{k_1, k_2}^{\alpha_1, \alpha_2} (B_{k_3, k_2}^{\alpha_3, \alpha_2})^* \right\rangle \left\langle B_{k_3, k_4}^{\alpha_3, \alpha_4} (B_{k_1, k_4}^{\alpha_1, \alpha_4})^* \right\rangle \\ & + \left\langle B_{k_1, k_2}^{\alpha_1, \alpha_2} (B_{k_1, k_4}^{\alpha_1, \alpha_4})^* \right\rangle \left\langle B_{k_3, k_2}^{\alpha_3, \alpha_2} (B_{k_3, k_4}^{\alpha_3, \alpha_4})^* \right\rangle. \end{aligned}$$

In the periodic-orbit theory, the SR and aSR pairs give the corresponding contribution to these terms. Let us consider the contribution from the first term

$$\begin{aligned} & \sum_{\alpha_1, \alpha_2} \sum_{k_1, k_2} \left(\frac{1}{2} \right)^2 i(2\hat{\epsilon}_{AC, k_1} - 2\hat{\epsilon}_{BD, k_2}) \\ & \times \left\{ \frac{1}{i(\hat{\epsilon}_{AC, k_1} - \hat{\epsilon}_{BD, k_2}) - 4s|\alpha_1 - \alpha_2|} \right\} \left\{ \frac{1}{i(\hat{\epsilon}_{AC, k_1} - \hat{\epsilon}_{BD, k_2}) - 4s|\alpha_1 + \alpha_2|} \right\}. \end{aligned} \quad (66)$$

This expression have 16 terms after taking the replica limit $r \rightarrow 0$, which is consistent with the the number of the SR pairs in the semiclassical theory. As easily checked, this reproduces the result in (38), if $8s$ is replaced by b . The second term generates the

contribution corresponding to that of the aSR pairs of the type in Figure 2(a):

$$\begin{aligned}
 & \sum_{\alpha_1, \alpha_2, \alpha_4} \sum_{k_1, k_2, k_4} \left(\frac{1}{2} \right)^2 \\
 & \times \left[i \{ 2\hat{\epsilon}_{AC, k_1} - (\hat{\epsilon}_{BD, k_2} + \hat{\epsilon}_{BD, k_4}) \} - 8s \left(\alpha_1 - \frac{\alpha_2 + \alpha_4}{2} \right)^2 \right] \\
 & \times \left\{ \frac{1}{i(\hat{\epsilon}_{AC, k_1} - \hat{\epsilon}_{BD, k_2}) - 8s \left| \frac{\alpha_1 - \alpha_2}{2} \right|} \right\} \left\{ \frac{1}{i(\hat{\epsilon}_{AC, k_1} - \hat{\epsilon}_{BD, k_4}) - 8s \left| \frac{\alpha_1 - \alpha_4}{2} \right|} \right\}.
 \end{aligned} \tag{67}$$

This has 64 terms after taking $r \rightarrow 0$. Finally the third term gives the contribution corresponding to that of the aSR pairs of the type in Figure 2(b):

$$\begin{aligned}
 & \sum_{\alpha_1, \alpha_3, \alpha_2} \sum_{k_1, k_3, k_2} \left(\frac{1}{2} \right)^2 \\
 & \times \left\{ i(\hat{\epsilon}_{AC, k_1} + \hat{\epsilon}_{AC, k_3} - 2\hat{\epsilon}_{BD, k_2}) - 8s \left(\frac{\alpha_1 + \alpha_3}{2} - \alpha_2 \right)^2 \right\} \\
 & \times \left\{ \frac{1}{i(\hat{\epsilon}_{AC, k_1} - \hat{\epsilon}_{BD, k_2}) - 8s \left| \frac{\alpha_1 - \alpha_2}{2} \right|} \right\} \left\{ \frac{1}{i(\hat{\epsilon}_{AC, k_3} - \hat{\epsilon}_{BD, k_2}) - 8s \left| \frac{\alpha_3 - \alpha_2}{2} \right|} \right\}.
 \end{aligned} \tag{68}$$

Eqs. (67) and (68) totally yield 128 terms, and reproduce the result in (44).

4.3. Relation between Semiclassical Formulas and Random Matrix Results

Let us consider how the semiclassical formulas are related to random matrix results. In the expansion (60), the factor \mathcal{F}_ℓ corresponds to the ℓ -encounter, while the V -th order terms in \mathcal{F}_ℓ are related to the terms with V encounters. The factors B and B^* represent the ports of the encounters. The contraction lines in the Wick theorem correspond to the links. Let us consider the factors $B_{k_i, k_{i+1}}^{\alpha_i, \alpha_{i+1}}$ and $(B_{k_{j+1}, k_j}^{\alpha_{j+1}, \alpha_j})^*$ in \mathcal{F}_ℓ and the corresponding ports. Note that α_{2l+1} and k_{2l+1} are identified with α_1 and k_1 , respectively. The ports are connected inside the ℓ -encounter as follows. If $i + 1 = j$, the original pseudo orbit connects the ports. On the other hand, if $i = j + 1$, the partner connects them. The superindex α on the left (right) side specifies the direction of the partner (original) pseudo orbit. The subindex k on the left (right) side determines the assignment of the index B or D (A or C). Using these rules, we can easily interpret Eq.(66)-(68) as the SR and aSR pairs.

As an example, let us examine the following case

$$\langle \overbrace{B_{k_1 k_2}^{\alpha_1 \alpha_2} (B_{k_3 k_2}^{\alpha_3 \alpha_2})^*} \overbrace{B_{k_3 k_4}^{\alpha_3 \alpha_4} (B_{k_1 k_4}^{\alpha_1 \alpha_4})^*} \overbrace{B_{k'_1 k'_2}^{\alpha'_1 \alpha'_2} (B_{k'_3 k'_2}^{\alpha'_3 \alpha'_2})^*} \overbrace{B_{k'_3 k'_4}^{\alpha'_3 \alpha'_4} (B_{k'_5 k'_4}^{\alpha'_5 \alpha'_4})^*} \overbrace{B_{k'_5 k'_6}^{\alpha'_5 \alpha'_6} (B_{k'_1 k'_6}^{\alpha'_1 \alpha'_6})^*} \rangle,$$

where the lines imply contractions. This example consists of two encounters with 2 and 3 stretches. In Figure 4, the solid curves depict the original pseudo orbit, and the

dashed curves depict the partner. The port B is located on the left side and B^* is on the right side of each encounter. In the figure, only the port indices and the connections among them are written. The thin curves (lines) express the connections inside the encounters, and the bold curves express the links generated by taking the contractions. Inside each encounter, the ports are connected by following the explained rules. The directions of the links and the encounter stretches are also shown in the figure. If the link connects B and B^* , then their left superindices (e.g., α_1 and α'_3) are the same and so are their right superindices (e.g., α_4 and α'_4). On the other hand, if the link connects B and B , or B^* and B^* , their left (right) superindices become the opposite. These properties originate from (62) and (63). The signs α can consistently be interpreted as the directions of the links. In Figure 4, the signs α are $\alpha_1 = \alpha_3 = \alpha'_3 = \alpha'_1 = -\alpha'_5 = +$, $\alpha_4 = -\alpha'_6 = -\alpha'_4 = -$, $\alpha'_2 = -$, and $\alpha_2 = +$.

In the semiclassical theory, for each topological structure of the pseudo-orbit pairs, we have to consider all the possible ways of assigning either A or C to the components the original pseudo orbit, and assigning either B or D to those of the partner. We see that each component accompanies a factor -1 , if C or D is assigned to it. In the replica treatment of random matrices, on the other hand, if C or D is assigned, the summation over all the possible choices of the indices leads to a factor $r - 1$, which goes to -1 in the limit $r \rightarrow 0$. Hence the sign finally becomes the same as that in the semiclassical formula.

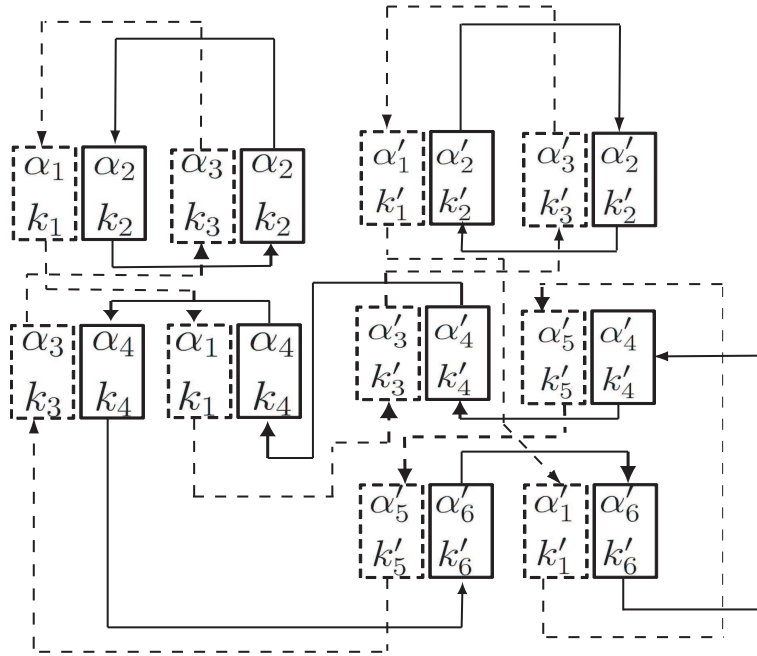


Figure 4. An example which consists of two encounters. Solid curves depict the original pseudo orbit, while the dashed curves depict the partner. The links correspond to the contractions in the Wick theorem. The directions of the links and the encounter stretches are also shown. In this case, $\alpha_1 = \alpha_3 = \alpha'_3 = \alpha'_1 = -\alpha'_5 = +$, $\alpha_4 = -\alpha'_6 = -\alpha'_4 = -$, $\alpha'_2 = -$, and $\alpha_2 = +$.

Finally, we discuss how many times the same structure appears, which gives the overcounting factor. Let us consider the ℓ -encounter term Eq.(64), and suppose that the contractions are already given.

First we change the indices as

$$(\alpha_{2m-1}, k_{2m-1}) \rightarrow (\alpha_{2m-1+p}, k_{2m-1+p}), \quad (69)$$

$$(\alpha_{2m}, k_{2m}) \rightarrow (\alpha_{2m+p}, k_{2m+p}) \quad (70)$$

for $m = 1, \dots, \ell$. For each value of $p = 1, \dots, \ell$, we find the same pseudo orbit pairs. This is related to the fact that we can arbitrarily choose one of the ℓ ports as the first.

Next we exchange the left and right ports in addition to the transformation

$$(\alpha_1, k_1) \rightarrow (\bar{\alpha}_3, k_3), \quad (71)$$

$$(\alpha_3, k_3) \rightarrow (\bar{\alpha}_1, k_1), \quad (72)$$

$$(\alpha_{m+3}, k_{m+3}) \rightarrow (\bar{\alpha}_{2\ell-1+m}, k_{2\ell+1-m}), \quad m \geq 1, \quad (73)$$

$$(\alpha_{2\ell+1-m}, k_{2\ell+1-m}) \rightarrow (\bar{\alpha}_{3+m}, k_{3+m}), \quad m \geq 1. \quad (74)$$

That is, changing the layout in Figure 4, we put the indices of B and B^* on the right and left sides, respectively. Then we recover the original pair of the pseudo orbits. This procedure yields two identical pairs. This related to the fact that we can relabel the left side of an encounter as the right side and vice versa in the semiclassical argument.

Thus we found that the overcounting factor was in total 2ℓ for each structure of the pseudo-orbit pairs.

4.4. Diagrammatic Expansion of the Generating Function

Now we are able to write down a diagrammatic expansion of \mathcal{Z}_{off} . Using (60) and the Wick theorem, we obtain a formal expression

$$\begin{aligned} \mathcal{Z}_{\text{off}} &= \sum_{V=1}^{\infty} \frac{1}{V!} \sum_{\ell_1, \ell_2, \dots, \ell_V} \left\langle \mathcal{F}_{\ell_1} \mathcal{F}_{\ell_2} \mathcal{F}_{\ell_3} \cdots \mathcal{F}_{\ell_V} \right\rangle \\ &= \lim_{r \rightarrow 0} \sum_{V=1}^{\infty} \frac{1}{V!} \sum_{\ell_1, \ell_2, \dots, \ell_V} \sum_{\text{contractions}} \sum_{k, \alpha}' \\ &\quad \times \prod_{j=1}^V \frac{(1/2\ell_j) \{i(m_{A_j} \epsilon_A - m_{B_j} \epsilon_B + m_{C_j} \epsilon_C - m_{D_j} \epsilon_D) - 8s n_{\text{enc},j}^2\}}{(-1) \{i(\epsilon_{A \text{ or } C} - \epsilon_{B \text{ or } D}) - 8s \mu\}}, \end{aligned} \quad (75)$$

where the symbol $\sum_{k, \alpha}'$ stands for the sum over all allowed k and α after taking the contractions. The corresponding values of m_{A_j}, \dots, m_{D_j} , $n_{\text{enc},j}$, μ and the assignments of $\epsilon_{A \text{ or } C}$, $\epsilon_{B \text{ or } D}$ are diagrammatically determined for each term. Note that a factor $1/2\ell_j$ is included in order to make up for the overcountings discussed above. We conjecture that this diagrammatic expansion is identified with the periodic-orbit expansion (47) derived in §3.

5. Summary

For a classically chaotic system in a weak magnetic field, the generating function of the spectral correlation was studied by using the periodic-orbit theory. The diagonal and leading off-diagonal terms in the periodic-orbit expansion were explicitly calculated. Both the non-oscillatory and oscillatory terms were in agreement with the prediction of the random matrix theory. Moreover we clarified the general structures of the higher order terms. In order to see the correspondence to the result of the random matrix theory, we analyzed the nonlinear sigma (NLS) model in the crossover domain between the GOE and GUE classes. It was conjectured that the diagrammatic expansion of the NLS model could be identified with the periodic-orbit expansion.

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Appendix A. Random Matrix Theory in GOE-GUE Crossover Domain

Appendix A.1. Derivation of the Nonlinear Sigma Model

In this Appendix, we derive the nonlinear sigma model in the crossover domain between the GOE and GUE classes of random matrices. The probability distribution of the random matrix H reads

$$P(H)dH = \mathcal{N} \exp \left[-\frac{N}{4 \{(\sigma_O - \sigma_U)e^{-2\tau} + \sigma_U\}} \left\{ \sum_{j=1}^N H_{jj}^2 + 2 \sum_{j<\ell} (\text{Re} H_{j\ell})^2 \right\} - \frac{N}{2\sigma_U (1 - e^{-2\tau})} \sum_{j<\ell} (\text{Im} H_{j\ell})^2 \right] dH, \quad (\text{A.1})$$

where \mathcal{N} is a normalization factor. Throughout this Appendix, we use the symbol \mathcal{N} for normalization factors, even if their values are different. The distribution (A.1) reproduces the GOE distribution P_{GOE} in the limit $\tau \rightarrow 0$, and the GUE distribution P_{GUE} in the limit $\tau \rightarrow \infty$, respectively:

$$\begin{aligned} P(H) &\rightarrow P_{\text{GOE}}(H) \\ &\propto \exp \left[-\frac{N}{4\sigma_O} \left\{ \sum_{j=1}^N H_{jj}^2 + 2 \sum_{j<\ell} (\text{Re} H_{j\ell})^2 \right\} \right] \prod_{j<\ell} \delta(\text{Im} H_{j\ell}), \quad \tau \rightarrow 0, \\ P(H) &\rightarrow P_{\text{GUE}}(H) \\ &\propto \exp \left[-\frac{N}{4\sigma_U} \left\{ \sum_{j=1}^N H_{jj}^2 + 2 \sum_{j<\ell} ((\text{Re} H_{j\ell})^2 + (\text{Im} H_{j\ell})^2) \right\} \right], \quad \tau \rightarrow \infty. \end{aligned}$$

We use the Bosonic replica trick to calculate the generating function defined by (1) and derive the nonlinear sigma model (NLS). Here we use the notation \mathcal{Z} to discriminate the NLS expression from the random matrix expression (3). The generating function is formally written in terms of the vectors $\psi_A, \psi_B, \psi_{C_k}$ and ψ_{D_k} ($k = 1, \dots, r-1$), each of which consists of N elements:

$$\begin{aligned} \mathcal{Z} &= \lim_{r \rightarrow 0} \int d[\psi] e^{i\psi^\dagger L E \psi} \\ &\times \left\langle \exp \left(-i\psi_A^\dagger H \psi_A - i \sum_{k=1}^{r-1} \psi_{C_k}^\dagger H \psi_{C_k} + i\psi_B^\dagger H \psi_B + i \sum_{k=1}^{r-1} \psi_{D_k}^\dagger H \psi_{D_k} \right) \right\rangle, \end{aligned} \quad (\text{A.2})$$

where ψ is a vector composed of ψ_A, ψ_{C_k} ($k = 1, \dots, r-1$), ψ_B , and ψ_{D_k} ($k = 1, \dots, r-1$). The symbol $\langle \dots \rangle$ denotes the random matrix average over the measure (A.1), and \mathbf{E} and L are $2rN \times 2rN$ matrices

$$\mathbf{E} = \text{diag}(E_A^+ \mathbf{1}, \overbrace{E_C^+ \mathbf{1}, \dots, E_C^+ \mathbf{1}}^{r-1}, E_B^- \mathbf{1}, \overbrace{E_D^- \mathbf{1}, \dots, E_D^- \mathbf{1}}^{r-1}), \quad (\text{A.3})$$

$$L = \text{diag}(\overbrace{\mathbf{1}, \mathbf{1}, \dots, \mathbf{1}}^r, \overbrace{-\mathbf{1}, -\mathbf{1}, \dots, -\mathbf{1}}^r). \quad (\text{A.4})$$

Here $\mathbf{1}$ is the $N \times N$ identity matrix. A random matrix average $\langle \dots \rangle$ can be easily evaluated, and we obtain

$$\mathcal{Z} = \lim_{r \rightarrow 0} \int d[\psi] e^{i\psi^\dagger L E \psi} \exp \left(-\frac{\lambda^2}{4N} \text{Tra}^2 + \frac{\mu^2}{4N} \text{Tr}b^2 \right), \quad (\text{A.5})$$

where $\lambda^2 = (\sigma_O - \sigma_U)e^{-2\tau} + \sigma_U$ and $\mu^2 = \sigma_U(1 - e^{-2\tau})$. The matrices a and b are $N \times N$ matrices, whose elements are given by

$$a_{\ell,m} = \sum_{\eta} \psi_{\eta,\ell}^* \Lambda^{\eta,\eta} \psi_{\eta,m} + \sum_{\eta} \psi_{\eta,m}^* \Lambda^{\eta,\eta} \psi_{\eta,\ell}, \quad (\text{A.6})$$

$$b_{\ell,m} = \sum_{\eta} \psi_{\eta,\ell}^* \Lambda^{\eta,\eta} \psi_{\eta,m} - \sum_{\eta} \psi_{\eta,m}^* \Lambda^{\eta,\eta} \psi_{\eta,\ell}. \quad (\text{A.7})$$

Here η is one of the indices $A, C_1, \dots, C_{r-1}, B, D_1, \dots, D_{r-1}$. Let us define $\Lambda^{\eta,\eta'}$ as the (η, η') element of the diagonal matrix

$$\Lambda = \text{diag}(\overbrace{1, 1, \dots, 1}^r, \overbrace{-1, -1, \dots, -1}^r). \quad (\text{A.8})$$

Next we introduce the vector Ψ_ℓ and $\bar{\Psi}_\ell$ which have $4r$ components as

$$\Psi_\ell = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_\ell \\ \psi_\ell^* \end{pmatrix}, \quad \bar{\Psi}_\ell = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_\ell^\dagger & \psi_\ell^T \end{pmatrix} \quad (\text{A.9})$$

Then Tra^2 and $\text{Tr}b^2$ can be written in terms of these vectors as

$$\text{Tra}^2 = 4 \sum_{\ell,m} \bar{\Psi}_\ell \mathbf{M} \Psi_m \bar{\Psi}_m \mathbf{M} \Psi_\ell, \quad (\text{A.10})$$

$$\text{Tr}b^2 = -4 \sum_{\ell,m} \bar{\Psi}_\ell \bar{\mathbf{M}} \Psi_m \bar{\Psi}_m \bar{\mathbf{M}} \Psi_\ell, \quad (\text{A.11})$$

where \mathbf{M} and $\bar{\mathbf{M}}$ are defined as

$$\mathbf{M} = \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \Lambda \end{pmatrix}, \quad \bar{\mathbf{M}} = \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & -\Lambda \end{pmatrix}. \quad (\text{A.12})$$

By introducing $4r \times 4r$ matrices A and \bar{A} as

$$A = \sum_{m=1}^N \Psi_m \bar{\Psi}_m \mathbf{M}, \quad \bar{A} = \sum_{m=1}^N \Psi_m \bar{\Psi}_m \bar{\mathbf{M}}, \quad (\text{A.13})$$

we can write the generating function in the form

$$\mathcal{Z} = \int d[\Psi] e^{i\psi^\dagger L \mathbf{E} \psi} e^{-\frac{\lambda^2}{N} (\text{Tr} A^2 + \nu^2 \text{Tr} \bar{A}^2)}, \quad (\text{A.14})$$

where $\nu^2 = (\mu/\lambda)^2$. Here and hereafter we omit the symbol $\lim_{r \rightarrow 0}$. In order to write A and \bar{A} in a unified way, we introduce \mathcal{A} as

$$\mathcal{A} = b_+ A + b_- \hat{\tau}_3 A \hat{\tau}_3, \quad (\text{A.15})$$

$$b_{\pm} = \frac{1}{2} \left(\sqrt{1 + \nu^2} \pm \sqrt{1 - \nu^2} \right), \quad (\text{A.16})$$

where $\hat{\tau}_3$ is

$$\hat{\tau}_3 = \text{diag}(\overbrace{1, 1, \dots, 1}^{2r}, \overbrace{-1, -1, \dots, -1}^{2r}). \quad (\text{A.17})$$

Notice that $\bar{A} = A \hat{\tau}_3$, and $\text{Tr} \mathcal{A}^2 = \text{Tr} A^2 + \nu^2 \text{Tr} [(A \hat{\tau}_3)^2]$. Thus, the generating function is written in terms of the new matrix \mathcal{A} as

$$\mathcal{Z} = \int d[\psi] e^{i\psi^\dagger L \mathbf{E} \psi} e^{-\frac{\lambda^2}{N} \text{Tr} \mathcal{A}^2}. \quad (\text{A.18})$$

We then employ the Hubbard-Stratonovich transformation to get rid of the term quadratic in A and consequently quartic in ψ . Using the relation $\mathcal{N} \int d[Q] \exp(\frac{N}{4} \text{Tr} Q^2) = 1$, where Q is anti-Hermitian, we can readily derive the expression

$$\mathcal{Z} = \mathcal{N} \int d[Q] \int d[\psi] \exp \left\{ i\psi^\dagger L \mathbf{E} \psi + \frac{N}{4} \text{Tr} Q^2 - \lambda \text{Tr}(\mathcal{A} Q) \right\}. \quad (\text{A.19})$$

Noting that the term $\text{Tr}(\mathcal{A} Q)$ is written as $\text{Tr}(\mathcal{A} Q) = \text{Tr}\{A(b_+ Q + b_- \hat{\tau}_3 Q \hat{\tau}_3)\}$. Hence, defining new matrix \tilde{Q} as

$$\tilde{Q} = b_+ Q + b_- \hat{\tau}_3 Q \hat{\tau}_3, \quad (\text{A.20})$$

we obtain

$$\mathcal{Z} = \mathcal{N} \int d[Q] \int d[\psi] \exp \left\{ i \sum_{m=1}^N \bar{\Psi}_m \mathbf{M} \left(\mathbf{F} + i\lambda \tilde{Q} \right) \Psi_m + \frac{N}{4} \text{Tr} Q^2 \right\}. \quad (\text{A.21})$$

Here \mathbf{F} is a $4r \times 4r$ matrix defined as

$$\mathbf{F} = \text{diag}(E_A, \overbrace{E_C, \dots, E_C}^{r-1}, E_B, \overbrace{E_D, \dots, E_D}^{r-1}, \\ E_A, \overbrace{E_C, \dots, E_C}^{r-1}, E_B, \overbrace{E_D, \dots, E_D}^{r-1}). \quad (\text{A.22})$$

In order to obtain Eq.(A.21), the relation $\psi^\dagger L \mathbf{E} \psi = \frac{1}{2} (\psi^\dagger L \mathbf{E} \psi + \psi^\dagger \mathbf{E} L \psi) = \frac{1}{2} (\psi^\dagger L \mathbf{E} \psi + \psi^T L \mathbf{E} \psi^*) = \sum_{m=1}^N \bar{\Psi}_m \mathbf{M} \mathbf{F} \Psi_m$ was used. Evaluating the ψ -integral, we reduce Eq.(A.21) to

$$\mathcal{Z} = \mathcal{N} \int d[Q] \exp \left\{ \frac{N}{4} \text{Tr} Q^2 - \frac{N}{2} \text{Tr} \ln(\mathbf{F} + i\lambda \tilde{Q}) \right\}. \quad (\text{A.23})$$

We set the average energy E to the origin (the band center), and write \mathbf{F} as

$$\mathbf{F} = \frac{1}{2\pi\bar{\rho}} \tilde{\epsilon}, \quad (\text{A.24})$$

$$\tilde{\epsilon} = \text{diag}(\epsilon_A, \overbrace{\epsilon_C, \dots, \epsilon_C}^{r-1}, \epsilon_B, \overbrace{\epsilon_D, \dots, \epsilon_D}^{r-1}, \epsilon_A, \overbrace{\epsilon_C, \dots, \epsilon_C}^{r-1}, \epsilon_B, \overbrace{\epsilon_D, \dots, \epsilon_D}^{r-1}), \quad (\text{A.25})$$

where the average density is $\bar{\rho} = N/(\pi\lambda)$. Now we need to consider the effect of ν . We treat it as a perturbation [20] and expand b_+ and b_- with respect to ν as $b_+ \sim 1$ and $b_- \sim \frac{\nu^2}{2}$. Setting $\nu = O(N^{-1/2})$, we find a dominant contribution to the generating function

$$\begin{aligned} \mathcal{Z} = \mathcal{N} \int d[Q] \exp \left\{ -\frac{N}{2} \text{Tr} \ln \left(1 + \frac{\frac{\lambda}{2N} \tilde{\epsilon} + \frac{i\lambda\nu^2}{2} \hat{\tau}_3 Q \hat{\tau}_3}{i\lambda Q} \right) \right\} \\ \times \exp \left[N \left\{ \frac{1}{4} \text{Tr} Q^2 - \frac{1}{2} \text{Tr} \ln(i\lambda Q) \right\} \right]. \end{aligned} \quad (\text{A.26})$$

The next step is the saddle point analysis in the limit $N \rightarrow \infty$. We find the saddle point equation

$$\frac{\delta}{\delta Q_{ij}} \left\{ \frac{1}{4} \text{Tr} Q^2 - \frac{1}{2} \text{Tr} \ln(i\lambda Q) \right\} = 0. \quad (\text{A.27})$$

This equation yields a saddle point Q_* satisfying

$$Q_*^2 = 1. \quad (\text{A.28})$$

As in the case of the GOE, the saddle point satisfies the symmetry relation

$$\sigma_1 Q_*^T \sigma_1 = Q_*. \quad (\text{A.29})$$

Here σ_1 is the Pauli matrix $\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$. Expanding the logarithmic part of Eq.(A.26), we obtain the nonlinear sigma model

$$\begin{aligned} \mathcal{Z} = \mathcal{N} \int d[Q_*] \exp \left\{ -\frac{N}{2} \text{Tr} \ln \left(1 + \frac{\frac{\lambda}{2N} \tilde{\epsilon} + \frac{i\lambda\nu^2}{2} \hat{\tau}_3 Q_* \hat{\tau}_3}{i\lambda Q_*} \right) \right\} \\ = \mathcal{N} \int d[Q_*] \exp \left[\frac{i}{4} \text{Tr}(\tilde{\epsilon} Q_*) - \frac{N\nu^2}{4} \text{Tr} \{ (Q_* \hat{\tau}_3)^2 \} \right] \\ = \mathcal{N} \int d[Q_*] \exp \left[\frac{i}{4} \text{Tr}(\tilde{\epsilon} Q_*) - \frac{s}{4} \text{Tr} \{ (Q_* \hat{\tau}_3)^2 \} \right]. \end{aligned} \quad (\text{A.30})$$

Here we introduced a parameter $s = N\nu^2$.

Appendix A.2. Parametrization

We further consider the exponential factor in Eq. (A.30). Let us denote the exponent in Eq.(A.30) by

$$\mathcal{S}(Q_*) = \frac{i}{4}\text{Tr}(\tilde{\epsilon}Q_*) - \frac{s}{4}\text{Tr}\{(Q_*\hat{\tau}_3)^2\}, \quad (\text{A.31})$$

As in the GOE case, we parametrize Q_* by using the matrices B and B^\dagger as

$$\begin{aligned} Q_* &= \left\{ \mathbf{1} - \begin{pmatrix} 0, & B \\ B^\dagger, & 0 \end{pmatrix} \right\} \begin{pmatrix} \mathbf{1}, & 0 \\ 0, & -\mathbf{1} \end{pmatrix} \left\{ \mathbf{1} - \begin{pmatrix} 0, & B \\ B^\dagger, & 0 \end{pmatrix} \right\}^{-1} \\ &= \left\{ \mathbf{1} - \begin{pmatrix} 0, & B \\ B^\dagger, & 0 \end{pmatrix} \right\} \begin{pmatrix} \mathbf{1}, & 0 \\ 0, & -\mathbf{1} \end{pmatrix} \sum_{n=0}^{\infty} \left(\begin{pmatrix} 0, & B \\ B^\dagger, & 0 \end{pmatrix} \right)^n. \end{aligned} \quad (\text{A.32})$$

In this representation, the symmetry relation of Q_s (A.29) is satisfied by requiring

$$B^\dagger = -\sigma_1 B^T \sigma_1. \quad (\text{A.33})$$

From a straightforward calculation, we obtain the expression of $\mathcal{S}(B)$ ($= \mathcal{S}(Q_*)$) in the form

$$\begin{aligned} \mathcal{S}(B) &= \frac{i}{2}(\epsilon_A - \epsilon_B - \epsilon_C + \epsilon_D) \\ &+ \frac{i}{2}\text{Tr}(\tilde{\epsilon}_{AC}BB^\dagger) - \frac{i}{2}\text{Tr}(\tilde{\epsilon}_{BD}B^\dagger B) - 2s\text{Tr}(BB^\dagger) + 2s\text{Tr}(B\tau_3B^\dagger\tau_3) \\ &+ \frac{i}{2} \sum_{m=2}^{\infty} [\text{Tr}\{\tilde{\epsilon}_{AC}(BB^\dagger)^m\} - \text{Tr}\{\tilde{\epsilon}_{BD}(B^\dagger B)^m\}] - 2s \sum_{m=2}^{\infty} \text{Tr}(BB^\dagger)^m \\ &- s \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \text{Tr}\{(BB^\dagger)^m \tau_3 (BB^\dagger)^{m'} \tau_3\} \\ &- s \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \text{Tr}\{(B^\dagger B)^m \tau_3 (B^\dagger B)^{m'} \tau_3\} \\ &+ 2s \sum_{m,m'=0;(m,m') \neq (0,0)}^{\infty} \text{Tr}\{(BB^\dagger)^m B \tau_3 (B^\dagger B)^{m'} B^\dagger \tau_3\}, \end{aligned} \quad (\text{A.34})$$

where $\tilde{\epsilon}_{AC,BD}$ and τ_3 are $2r \times 2r$ matrices defined in (50), (51) and (52).

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